

Glueball Spectrum for QCD from AdS Supergravity Duality*

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Abstract

We present the analysis of the complete glueball spectrum for the AdS^7 black hole supergravity dual of QCD_4 in strong coupling limit: $g^2N \rightarrow \infty$. The bosonic fields in the supergravity multiplet lead to 6 independent wave equations contributing to glueball states with $J^{PC} = 2^{++}, 1^{+-}, 1^{--}, 0^{++}$ and 0^{-+} . We study the spectral splitting and degeneracy patterns for both QCD_4 and QCD_3 . Despite the expected limitations of a leading order strong coupling approximation, the pattern of spins, parities and mass inequalities bare a striking resemblance to the known QCD_4 glueball spectrum as determined by lattice simulations at weak coupling.

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1 Introduction

The Maldacena duality conjecture [1] and its further extensions [2, 3] state that there is an exact equivalence between large N conformal field theories in d -dimensions and string theory in $\text{AdS}^{d+1} \times \mathbf{M}$. Subsequently Witten [4] suggested how to break explicitly the conformal (and SUSY) symmetries to arrive at a dual gravity description for $SU(N)$ quarkless QCD_4 . Thus we may have at last a definite proposal for the long sought “QCD string”. As anticipated by ’t Hooft the dual correspondence for the large $1/N$ expansion for $SU(N)$ Yang-Mills theory is the perturbative expansion of string theory. Still this theory is difficult to formulate, let alone solve. At present explicit calculations also require taking the strong coupling limit, $g^2 N \rightarrow \infty$, where the string tension goes to infinity ($\alpha' \rightarrow 0$) and the dual theory is classical gravity.

In this paper we complete the study of the glueball spectrum for the strong coupling dual description of QCD_4 . For comparison the analogous spectrum calculation is presented for QCD_3 , which shows a very similar pattern, which in qualitative terms can be traced to the underlying flat space T duality between type IIA and IIB string theories which in turn are the AdS duals to QCD_4 and QCD_3 respectively. The goal is to learn more about the AdS/Yang-Mills correspondence by comparing the AdS strong coupling spectrum with the rather well determined glueball spectrum [5] in lattice gauge theory. Of course, the strong coupling expansion at best can provide a rough guide to the underlying physics. Nonetheless the correspondence to the continuum (i.e. weak coupling) limit of the lattice spectrum is surprisingly good. This comparison may prove useful to provide support for the conjectured Maldacena duality and to give specific information on the major strong coupling artifacts that must be removed as one approaches universality at the ultraviolet fixed point.

In addition to new spectral calculations, we summarize the earlier work by many authors [6, 7, 8, 9, 10, 11, 12]. In particular, we extend our earlier paper [10] on the tensor glueball for QCD_3 on an AdS^5 black hole background to the physically relevant case of QCD_4 . For QCD_3 , we found that the tensor spectrum (2^{++}) was degenerate [14] with dilaton (0^{++}) and the axion (0^{+-}). However, the mass gap set by a lower scalar glueball obeys the inequality,

$$m(0^{++}) < m(2^{++}) . \tag{1}$$

This scenario is repeated for QCD_4 . The dilaton mode (0^{++}) coupling to $Tr[F^2]$ remains degenerate with the tensor (2^{++}), but the axion (0^{+-}) is heavier, consistent with lattice results. Due to the trace anomaly, the lowest mass scalar (0^{++}) couples to the energy density T_{00} and again it obeys this inequality above, Eq. (1). As in our earlier work,

the goal is to see the details of the spin structure of the lowest glueball states, which we believe is most sensitive to the underlying gauge theory. We find that our analysis combined with all the earlier results can give a systematic and complete strong coupling glueball spectrum.

It is useful to end this introduction with a rough overview of our results and the organization of the paper.

The geometrical construction for QCD_4 is roughly as follows. One starts with 11 dimensional M theory on $\mathbf{AdS}^7 \times \mathbf{S}^4$. The seven dimensional AdS^7 we take to have a radial co-ordinate r and Euclidean space-time co-ordinates x_1, x_2, x_3, x_4, x_5 and x_{11} . The “eleventh” dimension is taken to be compact, reducing the theory to type IIA string theory. Matter at the center of this space ($r = 0$) consists of N D4-branes (or NS 5-branes wrapping S^1 in the 11th coordinate) with world volume co-ordinates $x_1 \cdots x_5$. The 5-d Yang-Mills CFT “living on” the brane is dimensionally reduced to QCD_4 by raising the “temperature”, β^{-1} , in a direction $x_5 = \tau$, parallel to the brane. The new metric is an AdS^7 black hole with x_{11} compact. Compact directions on S^4 will be denoted by x_α , $\alpha = 7, 8, 9, 10$.

The strong coupling glueball calculation consists of finding the normal modes for the bosonic components of the supergraviton multiplet in the $\mathbf{AdS}^7 \times \mathbf{S}^4$ black hole background. We are only interested in excitations that lie in the superselection sector for QCD_4 . So we can ignore modes for all non-trivial harmonics in S^5 that carry a non-zero R charge and all Kaluza-Klein (KK) modes in the two S^1 circles with U(1) KK charges. Imposing these restriction and exploiting symmetries of the background metric reduce the problem to **six** independent wave equations, referred to as S_4, T_4, V_4, N_4, M_4 and L_4 in the text. In Fig. 1 (left side), we plot the low mass states for each equation, labeling the quantum numbers for each level. We can identify the modes with the bosonic components of the zero mass sector of type IIA string theory: the graviton (G), the dilaton (ϕ), the NS-NS 2-form (B) and RR 1- & 3-forms ($C_{(1)}, C_{(3)}$). The spin degeneracy of the spectrum is due to a spurious $O(4)$ symmetry of the strong coupling approximation that combines the 11th and three spatial co-ordinates. However, as we explain in the text, all extra states not observed in the lattice data for the glueball spectrum, actually carry a discrete “ τ -parity” that places them, like KK modes, outside the QCD superselection sector.

We have included in Fig. 1 (right side) the full spectrum for QCD_3 . The entire analysis is very similar. Starting with type IIB strings in $\mathbf{AdS}^5 \times \mathbf{S}^5$ which is dual to $\mathcal{N} = 4$ SUSY Yang-Mills, one introduces a compact thermal circle forming an AdS^5 black hole. Again

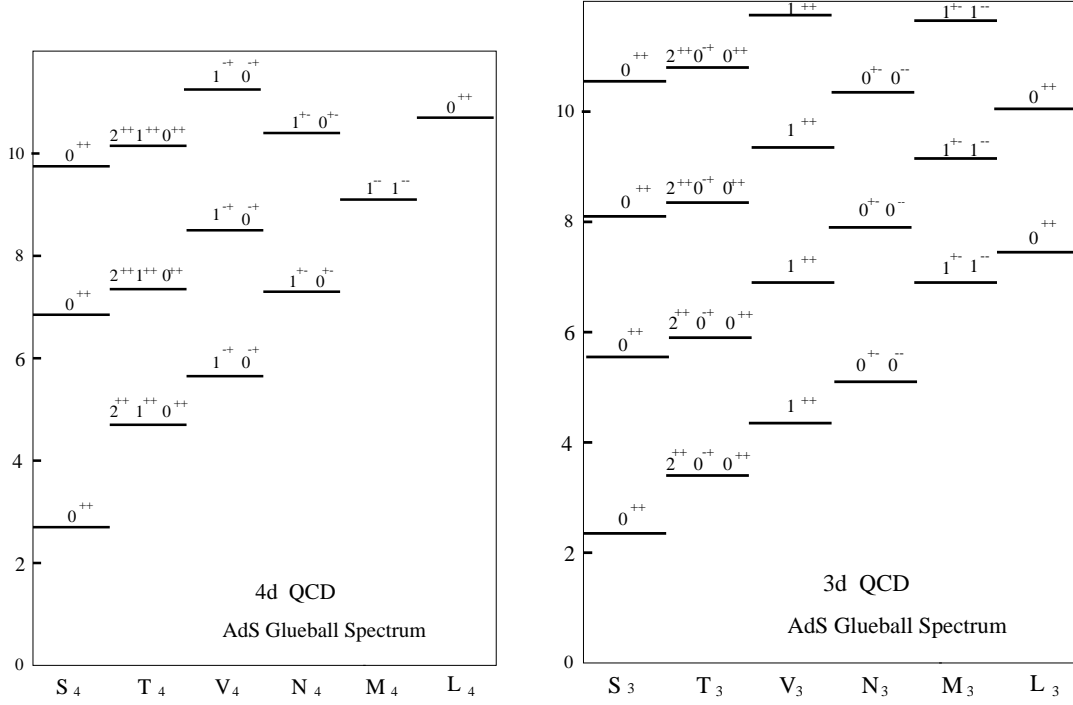


Figure 1: The AdS glueball spectrum for QCD_4 (left) and QCD_3 (right) with mass eigenvalues, m_n , plotted for each of the six equations labeled on the horizontal axis.

there are **six** independent wave equations, labeled by S_3, T_3, V_3, N_3, M_3 and L_3 . They correspond to fluctuations for type IIB fields: the graviton (G), the dilaton (ϕ), the NS-NS 2-form (B) as before and RR 0- & 2-forms ($C_{(0)}, C_{(2)}$). For both cases we also include volume fluctuations in the compact sphere S^4 and S^5 for type IIA and IIB respectively.

In Sec. 2, we give general arguments for the spin and degeneracy of glueball states for QCD_4 followed in Sec. 3 by the analysis for QCD_3 . For each, we give the resultant six wave equations and numerical values for the first ten levels. (Derivations for these equations are explained further in Appendix A.) For all but the lowest eigenvalue, the glueball masses, m_n , are well approximated by the WKB expansions [10, 13]: $m_n^2 = \mu^2(n^2 + \delta n + \gamma)$. For the first level ($n = 0$), we provide a simple but reasonably tight variational upper bound [10]. (See Appendix B for details.)

In Sec. 4, we give the parity and charge conjugation quantum numbers of the glueball

states using the Born-Infeld action to determine the quantum numbers of the couplings between gravity fields and gauge fields. The striking similarity between the QCD_4 and the QCD_3 spectra (see Fig. 1) can be understood qualitatively in terms of T-duality, which relates $D4$ branes in IIA to $D3$ branes in IIB.

In Sec. 5, we compare the AdS strong coupling spectrum with the well determined levels from lattice QCD_4 and remark on the relationship to the constituent gluon picture. No extra states are present in the AdS spectrum that couple to QCD operators, although the absence of the low mass 2^{-+} state is noted. We also show how the strong coupling expansion for the Pomeron intercept may be used to provide an estimate of the coupling at the crossover between the strong and weak coupling regimes.

2 Glueball Spectrum for QCD_4

To approach QCD_4 one begins with M theory on $AdS^7 \times S^4$. We compactify the “eleventh” dimension (on a circle of radius R_1) to reduce the theory to type IIA string theory and then following the suggestion of Witten raise the “temperature”, β^{-1} , with a second compact radius R_2 in a direction τ , with $\beta = 2\pi R_2$. On the second “thermal” circle, the fermionic modes have anti-periodic boundary conditions breaking conformal and all SUSY symmetries. This lifts the fermionic masses and also the scalar masses, through quantum corrections. The ’t Hooft coupling is $g^2 N = 2\pi g_s N l_s / R_2$, in terms of the closed string coupling, g_s and the string length, l_s . Therefore, in the scaling limit, $g^2 N \rightarrow 0$, if all goes as conjectured, there should be a fixed point mapping type IIA string theory onto $SU(N)$ pure Yang-Mills theory.

We consider the strong coupling limit at large N , where the string theory becomes classical gravity in the AdS^7 black hole metric,

$$ds^2 = (r^2 - \frac{1}{r^4})d\tau^2 + r^2 \sum_{i=1,2,3,4,11} dx_i^2 + (r^2 - \frac{1}{r^4})^{-1}dr^2 + \frac{1}{4}d\Omega_4^2, \quad (2)$$

with radius of curvature, $R_{AdS}^3 = 8\pi g_s N l_s^3$. We have removed all dimensionful parameters in the metric by a normalization setting $R_{AdS} = 1$ and $\beta = 2\pi/3$.

2.1 Spin and Degeneracy of Glueball States

In M theory the supergraviton is a single multiplet in 11-d with two bosonic fields - a graviton, G_{MN} , and a 3-form field, A_{MNL} , as designated in Table 1. After restricting all indices and co-ordinate dependence to AdS^7 , we have a graviton, $G_{\mu\nu}$, a dilaton ϕ , and an NS-NS tensor field $B_{\mu\nu}$. In addition there are two RR fields, a one-form C_μ and a three-form $C_{\mu\nu\lambda}$. Furthermore, we will also consider the scalar modes coming from “volume” fluctuations for S^4 . The relationship between M theory and IIA string theory nomenclature, after restricting to the AdS^7 subspace, is presented in Table 1. The table gives the J^{PC} quantum numbers for all glueball states. The pattern of degeneracy (explained below) is indicated by the rows ending with the lowest eigenvalue for each of the six wave equations, Eq. (9): T_4, V_4 , etc.

The task is to find all the quadratic fluctuations in the AdS^7 black hole background that might survive for QCD_4 in the scaling (weak coupling) limit, ignoring any Kaluza-Klein mode in compact manifolds (compactified S^1 for x_{11} , for τ and the spheres S^4). They are charge states in their own superselection sector that are clearly absent in the putative target theory. Additional “spurious” states will be discussed in Sec. 4 where we treat discrete symmetries.

States from 11-d G_{MN}				States from 11-d A_{MNL}		
$G_{\mu\nu}$	$G_{\mu,11}$	$G_{11,11}$	m_0 (Eq.)	$A_{\mu\nu,11}$	$A_{\mu\nu\rho}$	m_0 (Eq.)
G_{ij} 2^{++}	C_i $1^{++}_{(-)}$	ϕ 0^{++}	4.7007 (T_4)	B_{ij} 1^{+-}	C_{123} $0^{+-}_{(-)}$	7.3059 (N_4)
$G_{i\tau}$ $1^{+-}_{(-)}$	C_τ 0^{-+}		5.6555 (V_4)	$B_{i\tau}$ $1^{--}_{(-)}$	$C_{ij\tau}$ 1^{--}	9.1129 (M_4)
$G_{\tau\tau}$ 0^{++}			2.7034 (S_4)	G_α^α State 0^{++}		10.7239 (L_4)

Table 1: IIA Classification for QCD_4 . Subscripts to J^{PC} designate $P_\tau = -1$.

To count the number of independent fluctuations for a field of given spin, we adopt the following method. We imagine harmonic plane waves propagating in the AdS radial direction, r , with Euclidean time, x_4 . For example metric fluctuation,

$$G_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}(x), \quad (3)$$

in the background, $\bar{g}_{\mu\nu}$, are taken to have the form, $h_{\mu\nu}(r, x_4)$. There is no dependence on the spatial co-ordinates, $x_i = (x_1, x_2, x_3, x_{11})$ and the compactified “temperature” direction τ .

2.1.1 Metric fluctuations

The four dimensional field theory lives on the hypersurface co-ordinates, x_1, x_2, x_3, x_4 , (with $x_5 = \tau$ and x_{11} as the two compactified coordinates.) A graviton has two polarization indices. If we were in flat space time, we could go to a gauge where these indices took values only among $(x_1, x_2, x_3, x_{11}, \tau)$ and not from the set (r, x_4) . The polarization tensor should also be traceless. This leaves $(5 \times 6)/2 - 1 = 14$ independent components. In the AdS space time, we can count the number of graviton modes the same way, though the actual modes that we construct will have this form of polarization only at $r \rightarrow \infty$; for finite r , other components of the polarization will be constrained to acquire nonzero values [10].

We identify the spin content of these 14 components in two steps. First note that the background metric is $SO(4)$ symmetric. (It is flat in the first four of these directions, $\bar{g}_{11} = \bar{g}_{22} = \bar{g}_{33} = \bar{g}_{11,11} = r^2$, while it is “warped” in the τ direction, $\bar{g}_{\tau\tau} = r^2 - 1/r^4$). The system therefore has $SO(4)$ symmetry leading to *three distinct equations* corresponding to 9, 4 and 1 dimensional irreducible representations under $SO(4)$. In Table 1, these are denoted by T_4 , V_4 , and S_4 respectively.

These representations lead us to a degenerate spectrum of spins under the physical $SO(3)$ rotations in x_1, x_2, x_3 , which we list below:

- 9-dimensional representation breaks into $5 + 3 + 1$ under $SO(3)$,

$$\begin{aligned} G_{ij} : & \quad h_{ij} - \frac{1}{3}\delta_{ij}h_{kk} \neq 0 \quad \rightarrow \quad \text{spin-2}, \\ C_i : & \quad h_{i,11} = h_{11,i} \neq 0 \quad \rightarrow \quad \text{spin-1}, \\ \phi : & \quad h_{11,11} = -3h_{11} = -3h_{22} = -3h_{33} \neq 0 \quad \rightarrow \quad \text{spin-0}, \end{aligned} \quad (4)$$

- 4-dimensional representation breaks into $3 + 1$ under $SO(3)$,

$$\begin{aligned} G_{i\tau} : & \quad h_{\tau i} = h_{i\tau} \neq 0 \quad \rightarrow \quad \text{spin-1}, \\ C_\tau : & \quad h_{\tau,11} = h_{0\tau} \neq 0 \quad \rightarrow \quad \text{spin-0}, \end{aligned} \quad (5)$$

- singlet under $SO(3)$,

$$G_{\tau\tau} : \quad h_{\tau\tau} = -4h_{11} = -4h_{22} = -4h_{33} = -4h_{11,11} \neq 0 \quad \rightarrow \quad \text{spin-0}, \quad (6)$$

where $i, j, k = 1, 2, 3$.

In addition there is a scalar field, G_α^α , coming from the metric on the S^4 sphere [9], (with $m_{AdS}^2 = 72$), which is referred to as L_4 in Table 1.

2.1.2 Three-form fields

The behavior of the three-form field is discussed briefly in the Appendix, but we recall some of the main features here. The wave equation has a topological mass term which results in the equation being factorized into two first-order equations, yielding upon iteration two second-order equations. One solution is a massless 3-form field, which has solutions that are pure gauge for the case when there is no dependence on the sphere S^4 , and is thus to be ignored. The other field gives a second-order equation with $m_{AdS}^2 = 36$, but the fact that we have a first-order equation as the primary equation reduces the degrees of freedom effectively to those of a massless 3-form field. Let the propagation directions be again (x_4, r) . If we consider the component with indices A_{123} then we get a specific nonzero value also for the components $A_{r\tau,11}$ and $A_{4\tau,11}$. We can count the independent degrees of freedom by looking only at components that do not have the propagation directions x_4, r among the indices. Thus we get the fields listed in Table 1. Reducing the $SO(4)$ states under rotations in x_1, x_2, x_3 yields:

- 4-dimensional representation breaks into $3 + 1$ under $S0(3)$,

$$\begin{array}{ll} B_{ij} : & A_{ij,11} \neq 0 \rightarrow \text{spin-1,} \\ C_{123} : & A_{ijk} \neq 0 \rightarrow \text{spin-0,} \end{array} \quad (7)$$

- 6-dimensional representation into $3 + 3$ under $S0(3)$,

$$\begin{array}{ll} B_{i\tau} : & A_{i\tau,11} \neq 0 \rightarrow \text{spin-1 ,} \\ C_{ij\tau} : & A_{ij\tau} \neq 0 \rightarrow \text{spin-1 .} \end{array} \quad (8)$$

The field equations for these states have amplitudes N_4 and M_4 as listed in Table 1.

2.2 Wave Equations and QCD_4 Glueball Spectra

The wave equations for the metric fluctuations for QCD_4 have been obtained in Ref. [12] by analyzing the linearized Einstein equations about the $\mathbf{AdS}^7 \times \mathbf{S}^4$ black hole background

which leads to three independent equations, T_4 , V_4 and S_4 [10, 12]. Here we complete the spectral analysis giving the numerical values for all glueball masses. Fluctuations N_4 , M_4 and L_4 can be found similarly, leading again to all together six independent equations for QCD_4 , expressed in a manifestly hermitian form:

$$\begin{aligned}
& - \frac{d}{dr}(r^7 - r) \frac{d}{dr} T_4(r) - (m^2 r^3) T_4(r) = 0 , \\
& - \frac{d}{dr}(r^7 - r) \frac{d}{dr} V_4(r) - (m^2 r^3 - \frac{9}{r(r^6 - 1)}) V_4(r) = 0 , \\
& - \frac{d}{dr}(r^7 - r) \frac{d}{dr} S_4(r) - (m^2 r^3 + \frac{432r^5}{(5r^6 - 2)^2}) S_4(r) = 0 , \\
& - \frac{d}{dr}(r^7 - r) \frac{d}{dr} N_4(r) - (m^2 r^3 - 27r^5 + \frac{9}{r}) N_4(r) = 0 , \\
& - \frac{d}{dr}(r^7 - r) \frac{d}{dr} M_4(r) - (m^2 r^3 - 27r^5 - \frac{9r^5}{r^6 - 1}) M_4(r) = 0 , \\
& - \frac{d}{dr}(r^7 - r) \frac{d}{dr} L_4(r) - (m^2 r^3 - 72r^5) L_4(r) = 0 .
\end{aligned} \tag{9}$$

We shall provide a more detailed discussion on how these equations can be obtained in Appendix A, while concentrating here on establishing our normalization convention.

Consider first metric perturbations of the form

$$h_{\mu\nu} = \epsilon_{\mu\nu}(r) e^{ik_4 x_4} , \tag{10}$$

with all other fields set to zero. We shall further fix gauge to $h_{4\mu} = 0$, and from the linearized Einstein's equation, we determine the discrete spectrum with $k_4 = im$. Because of the $SO(4)$ symmetry in x_1, x_2, x_3, x_{11} , the system is highly degenerate. Three distinct equations for various perturbations can be obtained by the following procedure.

Tensor: There are five independent perturbations which form the spin-2 representations of $SO(3)$:

$$h_{ij} = q_{ij} r^2 T_4(r) e^{-mx_4} , \tag{11}$$

where $i, j = 1, 2, 3$ and q_{ij} is an arbitrary constant traceless-symmetric 3×3 matrix.

Vector: Consider perturbations:

$$h_{i\tau} = h_{\tau i} = q_i \sqrt{r^6 - 1} V_4(r) e^{-mx_4} , \tag{12}$$

where $i = 1, 2, 3$ and q_i is an arbitrary constant 3-vector. Both equations for T_4 and V_4 have also been obtained in Ref. [9] by considering the corresponding degenerate scalar modes.

Scalar: The analogous scalar perturbation is

$$h_{\tau\tau} = (r^2 - r^{-4})S_4(r) e^{-mx_4} . \quad (13)$$

Three-form and Volume Scalar: Next we turn to 3-form fields. It is sufficient to consider

$$\begin{aligned} A_{ij,11} = B_{ij} &= \epsilon_{ijk} q_k r^2 N_4(r) e^{-mx_4} , \\ A_{i\tau,11} = B_{i\tau} &= q_i \sqrt{r^6 - 1} M_4(r) e^{-mx_4} , \end{aligned} \quad (14)$$

where q_i is again an arbitrary constant 3-vector.

Note that the metric in the direction x_{11} is the same as that in the directions i, j, k , so the above functions N_4 , M_4 will also give the solutions for C_{ijk} and $C_{ij\tau}$ respectively. (Fluctuations for B_{ij} have been considered previously in Ref. [6].)

Lastly, for the volume perturbation, we consider [9]

$$h^\alpha_\alpha = L_4(r) e^{-mx_4} .$$

Equation: J^{PC} :	T_4 $2^{++}/1^{++}/0^{++}$	V_4 $1^{-+}/0^{-+}$	S_4 0^{++}	N_4 $1^{+-}/0^{+-}$	M_4 $1^{--}/1^{--}$	L_4 0^{++}
n = 0	22.097	31.985	7.308	53.376	83.046	115.002
n = 1	55.584	72.489	46.986	109.446	143.582	189.631
n = 2	102.456	126.174	94.485	177.231	217.399	277.282
n = 3	162.722	193.287	154.981	257.958	304.536	378.099
n = 4	236.400	273.575	228.777	351.895	405.018	492.169
n = 5	323.541	368.087	315.976	459.131	518.059	619.547
n = 6	424.195	474.268	416.666	579.706	646.088	760.252
n = 7	538.487	594.231	530.950	713.638	786.559	914.307
n = 8	666.479	729.102	658.996	860.939	939.557	1081.732
n = 9	808.398	875.315	800.860	1021.613	1108.010	1262.518

Table 2: The mass spectrum, m_n^2 , for QCD_4 Glueballs

To calculate the discrete spectrum for each of these equations, one must apply the correct boundary conditions at $r = 1$ and $r = \infty$. This issue has been discussed in several earlier

papers [7, 10]. The boundary conditions are found by solving the indicial equation. In all cases the appropriate boundary condition [7] at $r = 1$ is the one without the logarithmic singularity. At $r = \infty$ the least singular boundary is required to have a normalizable eigenstate. (See Appendix B for a listing of all boundary conditions.) Matching boundary conditions from $r = 1$ and $r = \infty$ results in a discrete set of eigenvalues m_n^2 , where n is the number of zeros in the wave function inside the interval $r \in (1, \infty)$. We solved the eigenvalue equations by the shooting method, integrating from $r_1 \simeq 1$ to large $r_\infty \simeq \infty$. The resultant spectrum is given in Table 2.

To further check our results, we have compared our numerical masses to the WKB approximations,

$$m_n^2 \simeq \mu_4^2(n^2 + \delta n + \gamma) + 0\left(\frac{1}{n}\right), \quad (15)$$

where $\mu_4^2 = 36\pi(\Gamma(2/3)/\Gamma(1/6))^2$. For each equation, individual integer constant, δ , was determined analytical and constant γ was fit to the numerical data in Table 4. (See Appendix B). The fits to the WKB formula are accurate to better than 0.1 % for all but the lowest ($n = 0$) mode. For each lowest mode, we have also carried out an independent simple variational estimate. We note that in case, our numerically calculated value for m_0^2 is always close and respects the variational estimate as an upper bound. (See Table 5.)

3 Glueball Spectrum for QCD_3

For QCD_3 the construction of the supergravity dual begins with Maldacena's conjecture for type IIB string theory in a $\mathbf{AdS}^5 \times \mathbf{S}^5$ background metric. Here the dual theory is conjectured to be the conformal field theory for $\mathcal{N} = 4$ SUSY SU(N) Yang-Mills in 4-d. The AdS curvature is induced by N units of charge on N coincident D3 branes giving rise to a constant volume 5-form $F_{(5)}$ in the product manifold. The co-ordinates in the AdS^5 space, we label by one "radial" co-ordinate r and four space-time co-ordinates, x_μ , $\mu = 1, 2, 3, 4$, parallel to the D3 branes. The remaining five co-ordinates in S^5 are labeled by x_α , $\alpha = 6, 7, 8, 9, 10$.

Following the suggestion of Witten for obtaining a supergravity dual to QCD_3 , we break conformal and SUSY symmetries, by introducing a compact "thermal" co-ordinate $x_4 = \tau$ with anti-periodic boundary on S^1 for the fermionic modes. The resultant metric at high temperature is an AdS^5 black hole,

$$ds^2 = \left(r^2 - \frac{1}{r^2}\right)d\tau^2 + r^2 \sum_{i=1,2,3} dx_i^2 + \left(r^2 - \frac{1}{r^2}\right)^{-1}dr^2 + d\Omega_5^2, \quad (16)$$

with radius of curvature, $R_{AdS}^4 = 4\pi g_s N l_s^4$ and 3-d Yang-Mills coupling, $g_3^2 N = 2g_s N/R$ in terms of the string coupling g_s , string length l_s and compact S^1 circumference $\beta = 2\pi R$. We have removed all dimensionful parameters from the metric by adopting a simple normalization with $R_{AdS} = 1$ and $\beta = \pi$, for the circumference of the thermal circle. At high temperature (or equivalently low energies), IIB string theory in this background is conjectured to be equivalent to QCD_3 .

3.1 Spin and Degeneracy of Glueball States

Type IIB string theory at low energy has a supergravity multiplet with several zero mass bosonic fields: a graviton, $G_{\mu\nu}$, a dilaton ϕ , an axion (or zero form RR field) C and two tensors, the NS-NS and RR fields $B_{\mu\nu}$ and $C_{\mu\nu}$ respectively. In addition there is the 4-form RR field $C_{(4)}$ that is constrained to have a self-dual field strength, $F_{(5)} = dC_{(4)}$.

Now the task is to find all the quadratic fluctuations in the above background metric whose eigen-modes correspond to the discrete glueball spectra for QCD_3 at strong coupling. We are only interested in excitations that lie in the superselection sector for QCD_3 . Thus for example we can ignore all non-trivial harmonic in S^5 that carry a non-zero R charge and all Kaluza-Klein (KK) modes in the S^1 thermal circle with a U(1) KK charge. The result of these considerations, discussed in detail below are summarized in Table 3.

$G_{\mu\nu}$	$e^{-\phi} + iC$ states		m_0 (Eq.)	$B_{\mu\nu}$	$C_{\mu\nu}$	m_0 (Eq.)
G_{ij} 2^{++}	C 0^{-+}	ϕ 0^{++}	3.4041 (T_3)	B_{ij} 0^{+-}	C_{ij} 0^{--}	5.1085 (N_3)
$G_{i\tau}$ 1^{++}			4.3217 (V_3)	$B_{i\tau}$ 1^{+-}	$C_{i\tau}$ 1^{--}	6.6537 (M_3)
$G_{\tau\tau}$ 0^{++}			2.3361 (S_3)	G_α^α State 0^{++}		7.4116 (L_3)

Table 3: IIB Classification for QCD_3

To count the number of independent fluctuations for a field of given spin, we again imagine harmonic plane waves propagating in the AdS radial direction, r , with Euclidean time, x_3 . For example, the metric fluctuations in AdS^5

$$G_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}(x), \quad (17)$$

in the fixed background $\bar{g}_{\mu\nu}$ are taken to be of the form $h_{\mu\nu}(r, x_3)$. There is no dependence on the spatial co-ordinates, $x_i = (x_1, x_2)$, or the compactified “temperature” direction, τ .

3.1.1 Metric fluctuations

A graviton has two polarization indices. If we were in flat space time, we could go to a gauge where these indices took values only among (x_1, x_2, τ) and not from the set (r, x_3) . The polarization tensor should also be traceless. This leaves $(3 \times 4)/2 - 1 = 5$ independent components. In the AdS space time, we can count the number of graviton modes the same way, though the actual modes that we construct will have this form of polarization only at $r \rightarrow \infty$; for finite r , other components of the polarization will be constrained to acquire nonzero values [10].

Therefore, a set of *five independent* polarization tensors can be characterized by the following non-vanishing components at $r \rightarrow \infty$. In the AdS^5 black hole background, τ is compact, so the rotations group is $SO(2)$ in (x_1, x_2) and the five states are in 3 irreducible representations: A spin-2 doublet (helicities ± 2), spanned by

$$G_{ij} : \quad h_{12} = h_{21} \neq 0, \quad \text{and} \quad h_{11} = -h_{22} \neq 0, \quad (18)$$

a spin-1 doublet (helicities ± 1), spanned by

$$G_{i\tau} : \quad h_{\tau 1} = h_{1\tau} \neq 0, \quad \text{and} \quad h_{\tau 2} = h_{2,\tau} \neq 0 \quad (19)$$

and a spin-0 state,

$$G_{\tau\tau} : \quad h_{\tau\tau} = -2h_{11} = -2h_{22} \neq 0. \quad (20)$$

These fluctuations are denoted by T_3 , V_3 , and S_3 respectively in Table 3.

3.1.2 Two-form fields

Each 2-form field in AdS^5 satisfies a field equation that includes a topological mass term. The 2-forms $B_{\mu\nu}$ and $C_{\mu\nu}$ can be combined into one complex 2-form field $\tilde{B}_{\mu\nu} = B_{\mu\nu} + iC_{\mu\nu}$. The field equation for $\tilde{B}_{\mu\nu}$ can be factorized into two first order equations, and each can be iterated leading to a second order equation of the form

$$\text{Max } \tilde{B}_{\mu\nu} + m_{AdS}^2 \tilde{B}_{\mu\nu} = 0,$$

where Max is the Maxwell operator on 2-forms. For modes that have no dependence on the S^5 co-ordinates, one equation is massive, with $m_{AdS}^2 = 16$, and the other is massless. It can be shown that the massless equation has only pure gauge solutions; so they can be ignored. (See Appendix A for further details.)

For the purpose of counting modes, polarizations for a massless 2-form in AdS^5 can also be restricted to be transverse, with fields depending only on (x_3, r) . Therefore the polarization tensor is an antisymmetric 2-tensor in the directions x_1, x_2, τ , leading to 3 independent components. On the other hand, for a general massive 2-form field, longitudinal polarizations are allowed, so that the polarization is an antisymmetric tensor in the coordinates x_1, x_2, τ, r , with 6 independent components.

Nevertheless, the number of independent components for our massive 2-form $\tilde{B}_{\mu\nu}$ is only 3 (complex), as if we are dealing with a massless case. This is due to the fact that the second order equation above stems from a first order equation, $\epsilon_{\mu\nu}^{\sigma\lambda\rho}\partial_{[\sigma}\tilde{B}_{\lambda\rho]} + 4i\tilde{B}_{\mu\nu} = 0$, relating real and imaginary parts and leading to additional constraints. For instance, with fields depending on (x_3, r) , if we start with the polarization tensor having $B_{12} = -B_{21} \neq 0$, then the first order equation will constrain us to have specific values for $C_{\tau r} = -C_{r\tau} \neq 0$ and $C_{3r} = -C_{r3} \neq 0$, while allowing all other components of B and C to be zero. Thus we count as independent fields $B_{12}, B_{1\tau}, B_{2\tau}, C_{12}, C_{1\tau}, C_{2\tau}$, i.e., there are three independent solutions for $B_{\mu\nu}$ and three solutions for $C_{\mu\nu}$.

The non-vanishing polarizations of the $B_{\mu\nu}$ tensor can be grouped as:

$$B_{ij} : \quad B_{12} = -B_{21} \neq 0, \quad (21)$$

corresponding to spin-0 and

$$B_{i\tau} : \quad \begin{aligned} B_{1\tau} &= -B_{\tau 1} \neq 0, \\ B_{2\tau} &= -B_{\tau 2} \neq 0. \end{aligned} \quad (22)$$

corresponding to a spin-1 doublet. In Table 3, these fluctuations are denoted by N_3 and M_3 respectively. They are degenerate with ones for the R-R 2-form $C_{\mu\nu}$.

3.1.3 Scalar fields

In general, for each scalar field, there is a unique field equation with the plane wave dependence which we have been considering. There are three such scalar modes: The volume fluctuations G_α^α in S^5 , (with $m_{AdS}^2 = 32$), denoted by L_3 [6], and fluctuations for

the dilaton ϕ and the axion C . However, as we have shown in an earlier paper [10], the latter two spectra are degenerate with the 2^{++} tensor fluctuations. Therefore, separate equations are not required [10, 11, 12].

3.2 Wave Equations and QCD_3 Glueball Spectrum

Metric fluctuations for QCD_3 have been obtained previously by analyzing the linearized Einstein equations about the $\mathbf{AdS}^5 \times \mathbf{S}^5$ black hole background which leads to three independent equations, T_3 , V_3 and S_3 [10, 11, 12]. Fluctuations N_3 , M_3 and L_3 can be found similarly, leading to all together six independent equations for QCD_3 . From the equation of motion, we determine the discrete spectrum with $k_3 = im$. (See Appendix A for details.) The full set of independent equations are:

$$\begin{aligned}
& - \frac{d}{dr}(r^5 - r) \frac{d}{dr} T_3(r) - (m^2 r) T_3(r) = 0 , \\
& - \frac{d}{dr}(r^5 - r) \frac{d}{dr} V_3(r) - (m^2 r - \frac{4}{r(r^4 - 1)}) V_3(r) = 0 . \\
& - \frac{d}{dr}(r^5 - r) \frac{d}{dr} S_3(r) - (m^2 r + \frac{64r^2}{(3r^4 - 1)^2}) S_3(r) = 0 , \\
& - \frac{d}{dr}(r^5 - r) \frac{d}{dr} N_3(r) - (m^2 r - 12r^3 + \frac{4}{r}) N_3(r) = 0 , \\
& - \frac{d}{dr}(r^5 - r) \frac{d}{dr} M_3(r) - (m^2 r - 12r^3 - \frac{4r^3}{r^4 - 1}) M_3(r) = 0 , \\
& - \frac{d}{dr}(r^5 - r) \frac{d}{dr} L_3(r) - (m^2 r - 32r^3) L_3(r) = 0 .
\end{aligned} \tag{23}$$

Each equation can be expressed in a variety of forms, depending on the choice of normalization. The following choices have been made so that each equation takes on a manifestly hermitian form:

Tensor:

$$h_{ij} = q_{ij} r^2 T_3(r) e^{-mx_3}, \tag{24}$$

where $i, j = 1, 2$, with q_{ij} an arbitrary constant traceless-symmetric 2×2 matrix.

Vector:

$$h_{i\tau} = q_i \sqrt{r^4 - 1} V_3(r) e^{-mx_3}, \tag{25}$$

where q_i is a constant 2-vector.

Equation: J^{PC} :	T_3 $2^{++}/0^{-+}/0^{++}$	V_3 1^{++}	S_3 0^{++}	N_3 $0^{+-}/0^{--}$	M_3 $1^{+-}/1^{--}$	L_3 0^{++}
n = 0	11.588	18.677	5.457	26.097	44.272	54.932
n = 1	34.527	47.499	30.442	61.159	84.318	100.628
n = 2	68.975	87.720	65.123	107.308	135.932	157.933
n = 3	114.910	139.436	111.141	164.829	199.073	226.773
n = 4	172.331	202.623	168.601	233.791	273.717	307.123
n = 5	241.237	277.283	237.528	314.215	359.861	398.971
n = 6	321.627	362.779	317.931	406.112	457.507	502.311
n = 7	413.501	461.121	409.815	509.486	566.631	617.140
n = 8	516.860	568.462	513.180	624.341	687.279	743.457
n = 9	631.703	689.156	628.028	750.677	819.329	881.260

Table 4: The mass spectrum, m_n^2 , for QCD_3 Glueballs

Scalar: This case has been treated carefully using several different gauge choices [10, 12]. Here we adopt the form suggested by Constable and Meyer [12] with

$$h_{\tau\tau} = (r^2 - r^{-2})S_3(r)e^{-mx_3}. \quad (26)$$

Two-forms and volume scalar: For B_{12} , consider perturbations of the form [6]

$$B_{12} = r^2 N_3(r) e^{ik_3 x_3}, \quad (27)$$

with $B_{1\tau} = B_{2\tau} = 0$. Alternatively, we consider

$$B_{i\tau} = q_i \sqrt{r^4 - 1} M_3(r) e^{-mx_3}, \quad (28)$$

with $B_{12} = 0$, where q_i is an arbitrary constant 2-vector. Lastly, for the volume perturbation, we consider

$$h_\alpha^\alpha = L_3(r) e^{-mx_3}.$$

To calculate the discrete spectrum for each of these equations one must again apply the correct boundary conditions at $r = 1$ and $r = \infty$ as mentioned in the case of QCD_4 earlier. (See Appendix B for a listing of boundary conditions.) The resultant spectrum is given in Table 4.

Similarly, we have also compared our numerical QCD_3 masses to the WKB approximations,

$$m_n^2 \simeq \mu_3^2(n^2 + \delta n + \gamma) + 0\left(\frac{1}{n}\right), \quad (29)$$

where $\mu_3^2 = 16\pi(\Gamma(3/4)/\Gamma(1/4))^2$, with integer constants, δ , listed in Table 6, determined analytically. Again, the constants γ were fits to the numerical data. (See Appendix B).

4 Parity and Charge Conjugation assignments

Next we determine how the supergravity fields and therefore the glueballs couple to the boundary gauge theory. This allows us to unambiguously assign the correct parity and charge quantum numbers to the glueball states. For this purpose we consider the effective Born-Infeld action on the branes.

4.1 QCD_4

The 4-d gauge theory for QCD_4 is obtained by dimensional reduction from a 5-d gauge theory, which is the low energy dynamics of D4-branes in 10-dimensional Type IIA string theory. Although this 10-d theory may itself be regarded as a dimensional reduction of 11-d M-theory for membranes, it is sufficient and more convenient to consider the 10-d theory itself.

Since supergravity fields can be thought of as coupling constants for gauge theory operators, their quantum numbers can be assigned by the parity and charge conjugation invariance of the overall action, (supergravity field times composite operator). For simplicity let us consider the coupling of a supergravity field to just one D4-brane — this coupling is given by a Born-Infeld action plus a Wess-Zumino term,

$$S = \int d^5x \det[G_{\mu\nu} + e^{-\phi/2}(B_{\mu\nu} + F_{\mu\nu})] + \int d^4x (C_1 F \wedge F + C_3 \wedge F + C_5), \quad (30)$$

where $\mu, \nu = 1, 2, 3, 4, \tau$. Later we will argue that our quantum number assignment is correct also for the non-abelian case of N coincident D branes. In the 5-d field theory, we have the space-time world volume co-ordinate x_1, x_2, x_3, x_4, τ with τ compactified on S^1 . The Euclidean time coordinate we take to be x_4 . After dimensional reduction the physical fields will be characterized by their representation under the little group $SO(3)$ of rotations on the spatial co-ordinates x_i , $i = 1, 2, 3$, in the 4-d theory.

For the 5-d gauge fields, we define parity by

$$P : A_i(x_i, x_4, \tau) \rightarrow -A_i(-x_i, x_4, \tau),$$

$$\begin{aligned}
P & : A_4(x_i, x_4, \tau) \rightarrow A_4(-x_i, x_4, \tau), \\
P & : A_\tau(x_i, x_4, \tau) \rightarrow A_\tau(-x_i, x_4, \tau),
\end{aligned} \tag{31}$$

for $x_i \rightarrow -x_i$, $x_4 \rightarrow x_4$, and $\tau \rightarrow \tau$. For the Euclidean \mathbf{R}^5 space, this is the only discrete symmetry. However after compactification to $\mathbf{R}^4 \times \mathbf{S}^1$, we can define another parity (not related by a 5-d proper Lorentz transformation) by inverting the τ co-ordinate on \mathbf{S}^1 . Thus we define a separate discrete τ -parity transformation $P_\tau : \tau \rightarrow -\tau$,

$$\begin{aligned}
P_\tau & : A_i(x_i, x_4, \tau) \rightarrow A_i(x_i, x_4, -\tau), \\
P_\tau & : A_4(x_i, x_4, \tau) \rightarrow A_4(x_i, x_4, -\tau), \\
P_\tau & : A_\tau(x_i, x_4, \tau) \rightarrow -A_\tau(x_i, x_4, -\tau).
\end{aligned} \tag{32}$$

Charge conjugation for a non-abelian gluon field is

$$C : \frac{1}{2}T_a A_\mu^a(x) \rightarrow -\frac{1}{2}T_a^* A_\mu^a(x) \tag{33}$$

where T^a are the Hermitian generators of the group. In terms of matrix fields ($A \equiv \frac{1}{2}T_a A^a$), $C : A_\mu(x) \rightarrow -A_\mu^T(x)$. This leads to a subtlety. For example consider the transformation of a trilinear gauge invariant operators,

$$C : Tr[F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}] \rightarrow -Tr[F_{\mu_3\nu_3}F_{\mu_2\nu_2}F_{\mu_1\nu_1}]. \tag{34}$$

The order of the fields is reversed. Hence the symmetric products, $d^{abc}F_1^a F_2^b F_3^c$, have $C = -1$ and the antisymmetric products, $f^{abc}F_1^a F_2^b F_3^c$, $C = +1$. Of course using a single brane, we can only find symmetric products. For reasons explained further in Sec. 5, we will only encounter symmetric traces over polynomials in F , designate by $Sym \text{Tr}[F_{\mu\nu} \dots]$. Even polynomials have $C = +1$ and odd polynomials $C = -1$.

4.1.1 Graviton couplings

Expanding the Born-Infeld action, we can now read off the $J^{PC}(P_\tau)$ assignments:

From the coupling, $G_{\mu\nu}T^{\mu\nu} \sim G_{\mu\nu}\text{Tr}(F_{\mu\lambda}F_\nu^\lambda) + \dots$, we obtain

$$G_{ij} \rightarrow 2^{++} (P_\tau = +), \quad G_{i\tau} \rightarrow 1^{-+} (P_\tau = +) \quad G_{\tau\tau} \rightarrow 0^{++} (P_\tau = +). \tag{35}$$

Under compactification of 11-d supergravity theory, $G_{\mu,11}$ becomes the Ramond-Ramond 1-form C_μ , which couples as $\sim \epsilon^{\mu\nu\lambda\kappa\eta} C_\mu \text{Sym Tr}[F_{\nu\lambda}F_{\kappa\eta}W]$, where W is an even power of fields F . Consequently, the coupling, $\epsilon^{ijk} C_i \text{Sym Tr}[F_{\tau j}F_{k4}W] + \dots$ leads to

$$C_i \rightarrow 1^{++} (P_\tau = -). \tag{36}$$

Similarly, $\epsilon^{ijk}C_\tau \text{Tr}(F_{ij}F_{4k}W) + \dots$ gives

$$C_\tau \rightarrow 0^{-+} \quad (P_\tau = +) , \quad (37)$$

and $G_{11,11}$ leads to the dilaton ϕ with coupling $\phi \text{Tr}F^2$,

$$\phi \rightarrow 0^{++} \quad (P_\tau = +) . \quad (38)$$

4.1.2 Two-form, three-form fields, and volume scalar

Consider first the NS-NS 2-form field $B_{\mu\nu}$. This field in the Type IIA theory arises from the 3-form field of the 11-d supergravity theory when the components of the 3-form field are $A_{\mu\nu,11}$. For $U(1)$ gauge theory in leading order this field couples as $B_{\mu\nu}F^{\mu\nu}$. More generally in the $SU(N)$ gauge theory, we must have a multi-gluon coupling, $B_{\mu\nu} \text{SymTr}[F_{\mu\nu}W]$. (Again W is an even power of fields F and the trace is symmetrized.)

To determine the parity, assume for the supergravity modes that we are in a gauge where the indices of the 2-form do not point along x_4, x_{11}, r . With $i, j = 1, 2, 3$, this leads to coupling $B_{ij} \text{SymTr}[F^{ij}W]$ with

$$B_{ij} \rightarrow 1^{+-} \quad (P_\tau = +) , \quad (39)$$

and coupling $B_{i\tau} \text{SymTr}[F^{i\tau}W]$ with

$$B_{i\tau} \rightarrow 1^{--} \quad (P_\tau = -) . \quad (40)$$

An analogous analysis can also be carried out for 3-form fields, C_{ijk} and $C_{ij\tau}$. The coupling, $C_{123} \text{SymTr}[F^{4\tau}W]$ leads to

$$C_{ijk} \rightarrow 0^{+-} \quad (P_\tau = -) , \quad (41)$$

and coupling $\epsilon^{ijk}C_{ij\tau} \text{SymTr}[F^{4k}W]$ leads to

$$C_{ij\tau} \rightarrow 1^{--} \quad (P_\tau = +) . \quad (42)$$

Lastly, the volume scalar couples as $h_\alpha^\alpha \text{SymTr}F^4 + \dots$ giving

$$h_\alpha^\alpha \rightarrow 0^{++} \quad (P_\tau = +) . \quad (43)$$

The complete parity and charge conjugation assignments are given in Table 1.

4.2 QCD_3

The 3-d gauge theory is obtained by dimensional reduction from a 4-d gauge theory. To find the symmetries of the interactions, we consider the Born-Infeld action plus Wess-Zumino term, describing the coupling of a supergravity field to a single D3-brane,

$$S = \int d^4x \det[G_{\mu\nu} + e^{-\phi/2}(B_{\mu\nu} + F_{\mu\nu})] + \int d^4x (C_0 F \wedge F + C_2 \wedge F + C_4) ,$$

where $\mu, \nu = 1, 2, 3, \tau$. As in the case of QCD_4 , we will find the charge conjugation and parity assignments with the help of the symmetries of the 4-d gauge theory and then take the dimensional reduction to the 3-d theory after compactification of the coordinate τ . The Euclidean time is taken to be x_3 and the spatial co-ordinates, x_i , $i = 1, 2$.

After dimensional reduction the physical fields will be characterized by their representation under the little group of the space co-ordinates of the 3-d theory: this is the group $SO(2)$ rotations in the x_1, x_2 plane. However, unlike the case of QCD_4 , the usual spatial inversion, $x_i \rightarrow -x_i$ with $x_3 \rightarrow x_3$ and $\tau \rightarrow \tau$, is a rotation $R(\theta)$ in $SO(2)$ with $\theta = \pi$; it therefore does not lead to a discrete symmetry. A discrete symmetry can be defined by $x_1 \rightarrow x_1$ and $x_2 \rightarrow -x_2$, as was pointed out in the lattice studies by Harte and Philipsen [15]. However, the sole manifestation of this symmetry is the helicity “doublets”, $\lambda = \pm J$, for states with spin $J > 0$. We have already taken this degeneracy into account.

On the other hand, τ -parity remains a discrete symmetry of the action, as is the case for QCD_4 . In this paper, we shall define “parity” for QCD_3 as $P \equiv R(\pi) \times P_\tau$ where $x_i \rightarrow -x_i$, $x_3 \rightarrow x_3$, and $\tau \rightarrow -\tau$. This of course is precisely the parity for the uncompactified 4-d theory with x_3 treated as Euclidean time.

4.2.1 Graviton, dilaton and axion states

The graviton $G_{\mu\nu}$ couples as $G_{\mu\nu}T^{\mu\nu}$ as in the case of QCD_4 . Because an even number of gluons occur in the field operators, the charge conjugation for all such states are $C = +$. For parity, we assume we are in a gauge where the indices of $G_{\mu\nu}$ do not point along x_3, τ . From the coupling, $G_{\mu\nu}\text{Tr}[F^{\mu\lambda}F_\lambda^\nu] + \dots$, we get states

$$G_{ij} \rightarrow 2^{++} \ (P_\tau = +), \quad G_{i\tau} \rightarrow 1^{++} \ (P_\tau = -), \quad G_{\tau\tau} \rightarrow 0^{++} \ (P_\tau = +). \quad (44)$$

The dilaton couples as $\phi\text{Tr}F^2$, leading to

$$\phi \rightarrow 0^{++} \ (P_\tau = +), \quad (45)$$

and for an axion coupling $C_0 \text{Tr}(F_{12}F_{\tau 3})$,

$$C_0 \rightarrow 0^{-+} \quad (P_\tau = -) . \quad (46)$$

4.2.2 Two-form fields and volume scalar

Consider first the NS-NS 2-form field $B_{\mu\nu}$. For parity, again assume that we are in a gauge where the indices of the 2-form do not point along x_3, r . With $i, j = 1, 2$, the coupling $B_{ij} \text{SymTr}[F^{ij}W]$ leads to

$$B_{ij} \rightarrow 0^{+-} \quad (P_\tau = +) , \quad (47)$$

and $B_{i\tau} \text{SymTr}[F^{i\tau}W]$ to

$$B_{i\tau} \rightarrow 1^{+-} \quad (P_\tau = -) . \quad (48)$$

Finally for the Ramond-Ramond 2-form $C_{\mu\nu}$, we have the coupling $C_{12} \text{SymTr}[F_{\tau 3}W]$, so

$$C_{12} \rightarrow 0^{--} \quad (P_\tau = -) , \quad (49)$$

and $\epsilon^{ij}C_{\tau i} \text{SymTr}[F_{j3}W]$ giving

$$C_{i\tau} \rightarrow 1^{--} \quad (P_\tau = +) . \quad (50)$$

Finally, as in the case of QCD_4 , the volume scalar couples as $h_\alpha^\alpha \text{Tr}F^4 + \dots$ so that

$$h_\alpha^\alpha \rightarrow 0^{++} \quad (P_\tau = +) . \quad (51)$$

The complete parity and charge conjugation assignments are given in Table 3.

5 Discussion

Lastly we turn to the question of how well the strong coupling limit for the Maldacena dual theory of QCD represents the infrared physics probed by the glueball spectra. Happily we now have a rather definitive lattice glueball spectrum by Morningstar and Pearson [5] with which to make comparisons (See right side of Fig. 2 below).

5.1 Comparison with Lattice Glueball Spectrum

Originally, claims were made about accurate comparisons to a few percent for isolated (scalar) mass ratios. As we pointed out in Ref. [10] for QCD_3 , the lowest mass scalar comes from the gravitational multiplet, not the dilaton. A similar spectrum is observed for QCD_4 . Consequently such accurate mass ratios were a misconception. This should not be regarded as a failure, since any reasonable expectation of a strong coupling approximation should not give quantitative results. On the other hand, there is a rather remarkable correspondence of the overall mass and spin structure between our strong coupling glueball spectrum and the lattice results at weak coupling for QCD_4 (see Fig. 2 below.) Apparently the spin structure of type IIA supergravity does resemble the low mass glueball spin splitting. The correspondence is sufficient to suggest that the Maldacena duality conjecture may well be correct and that further efforts to go beyond strong coupling are worthy of sustained effort.

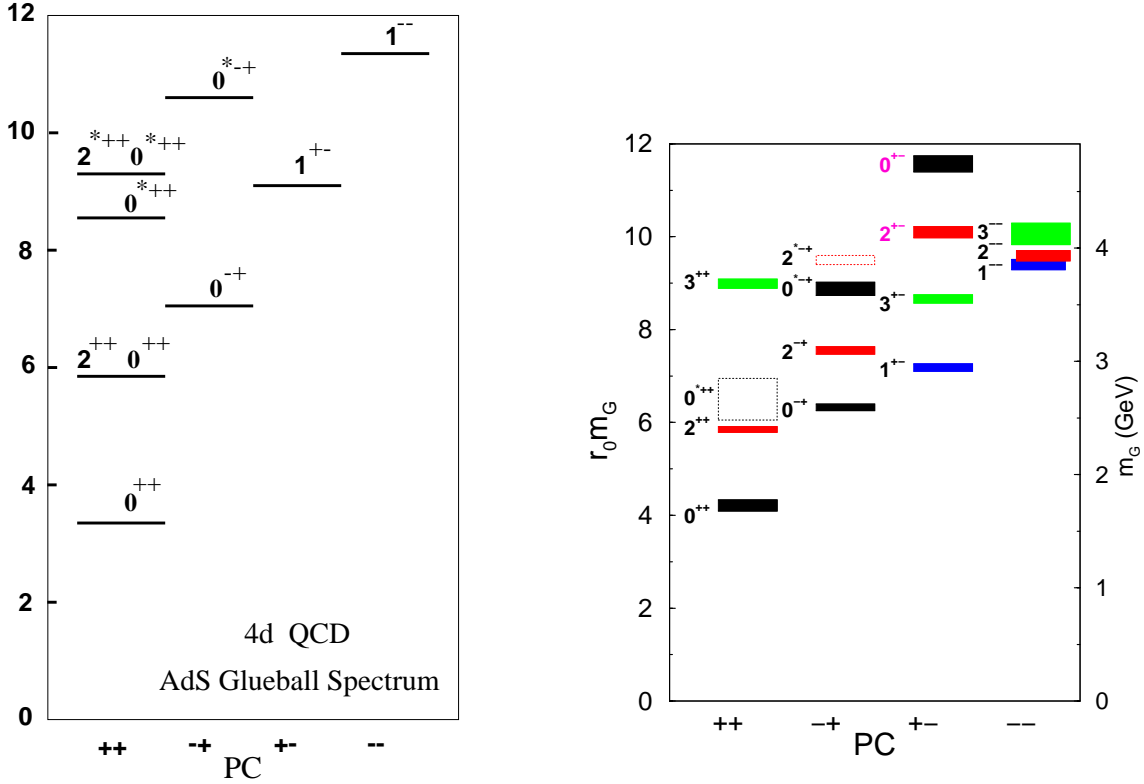


Figure 2: The AdS glueball spectrum for QCD_4 in strong coupling (left) compared with the lattice spectrum [5] for pure $SU(3)$ QCD (right). The AdS cut-off scale is adjusted to set the lowest 2^{++} tensor state to the lattice results in units of the hadronic scale $1/r_0 = 410$ Mev.

Note that for each value of $PC = (++, +-, +-, --)$, the lowest state is present in approximately the right mass range. In addition, the exact $2^{++}/0^{++}$ degeneracy for AdS strong coupling corresponds to a relatively small splitting in the lattice calculations. Finally, there is a radial excitation of the pseudoscalar 0^{*-+} that suggests that even this effect is approximated. This is intriguing because in the supergravity description the radial mode is a standing wave in the extra 5th dimension whereas in the lattice it is a conventional radial mode. Apparently scale changes in 4-d are being represented by the distance into the extra “warped” 5th axis.

At higher masses the discrepancies increase. One reason is the obvious fact that on the supergravity side all orbital excitations of higher spin states are pushed to infinity in strong coupling by virtue of the divergent string tension,

$$\sigma \equiv \frac{1}{2\pi\alpha'} = \frac{16\pi g^2 N}{27} \left[1 + 0\left(\frac{1}{g^2 N}\right) \right]. \quad (52)$$

For example, the 3^{++} state is a purely stringy effect outside of the classical limit of supergravity.

Finally, we must emphasize that our comparison is premised on the neglect of many “spurious” states in the strong coupling limit that are in the wrong superselection sector to survive in the conjectured weak coupling limit of QCD. For example, all the Kaluza-Klein modes in the compact thermal S^1 manifold and the sphere S^4 (or S^5) have masses at the cut-off scale. (The first mode on the thermal circle has a KK mass scale $m_{KK}^2 = 4$ in type IIB and $m_{KK}^2 = 9$ in type IIA in the units used in Table 4 and 2 respectively.) But these spurious KK modes all carry conserved $U(1)$ or R charges that are absent in the target theory. We assume they will disappear in the continuum limit. A subtler situation occurs in the QCD_4 example. Because normal modes in the extreme strong coupling limit do not distinguish between the compact 11th dimension and the spatial co-ordinates x_1, x_2, x_3 on the brane, the spectrum actually has an exact $SO(4)$ symmetry. Thus there are additional states (listed in Table 1 but ignored in Fig. 2) exactly degenerate with the physically reasonable $2^{++}/0^{++}$, 0^{-+} , 1^{+-} and 1^{--} states. They are all odd under the discrete symmetry of reflecting the thermal circle (i.e. $P_\tau = -$) so they also lie in another superselection sector. A major challenge is to understand how this $SO(4)$ symmetry is lifted and if the unwanted states remain at the cut-off in weak coupling. Physically the 11th axis is very different. The membranes of 11-d M-theory wraps this axis. Another possibility worth exploring is modifying the background metric with an orbifold that projects directly onto the even $\tau-$ parity sector for QCD.

5.2 Constituent Gluon Picture

The basic idea behind the AdS/CFT correspondence in the context of glueballs is similar to an observation made much earlier by Fritzsche and Minkowski [22], by Bjorken [23] and by Jaffe, Johnson and Ryzak [24]. Namely that the low mass glueball spectrum can be qualitatively understood in terms of local gluon interpolating operators of minimal dimension.

For example, Ref. [24] lists all gauge invariant operators for dimension $d = 4, 5$ and 6. Eliminating operators that are zero by the classical equation of motion and states that decouple because of the conservation of the energy momentum tensor, the operators are in rough correspondence with all the low mass glueballs states, as computed in a constituent gluon [25] or bag model. Indeed more recently Kuti [26] has pointed out that a more careful use of the spherical cavity approximation even gives a rather good quantitative match to the lowest 11 states in the lattice spectrum.

Consequently it is interesting to compare this set of operators with the supergravity model. We list below all the operators for $d \leq 6$ except the operators with explicit derivatives (e.g. $Tr[FDF]$) and $Tr[FDDF]$):

Dimension	State	Operator	Supergravity
d = 4	0^{++}	$Tr(FF) = \vec{E}^a \cdot \vec{E}^a - \vec{B}^a \cdot \vec{B}^a$	ϕ
d = 4	2^{++}	$T_{ij} = E_i^a \cdot E_j^a + B_i^a \cdot B_j^a - \text{trace}$	G_{ij}
d = 4	0^{-+}	$Tr(F\tilde{F}) = \vec{E}^a \cdot \vec{B}^a$	C_τ
d = 4	0^{++}	$2 T_{00} = \vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a$	$G_{\tau\tau}$
d = 4	2^{-+}	$E_i^a \cdot B_j^a + B_i^a \cdot E_j^a - \text{trace}$	Absent
d = 4	2^{++}	$E_i^a \cdot E_j^a - B_i^a \cdot B_j^a - \text{trace}$	Absent
d = 6	$(1, 2, 3)^{\pm-}$	$Tr(F_{\mu\nu}\{F_{\rho\sigma}, F_{\lambda\eta}\}) \sim d^{abc} F^a F^b F^c$	$B_{ij}, C_{ij\tau}$
d = 6	$(0, 1, 2)^{\pm+}$	$Tr(F_{\mu\nu}[F_{\rho\sigma}, F_{\lambda\eta}]) \sim f^{abc} F^a F^b F^c$	Absent

In this table we have used a Minkowski metric. The classification is parallel to our discussion of couplings in the Born-Infeld action for D4 branes (see Sec. 4.1), except that all components and derivatives in the 5th, i.e., τ , direction are zero and therefore all $P_\tau = -1$ states are absent (See Table 1). The column on the right lists the supergravity mode that couples (after dropping τ components) to each operator.

Several observations are in order. For the d = 4 operators, there is complete agreement

on the 3 lowest quantum numbers: $J^{PC} = 0^{++}, 2^{++}, 0^{-+}$. Also there is agreement on the absence of a low mass 1^{-+} state that Ref. [24] attributes to the conservation law that decouples the operator corresponding to the momentum tensor $T_{0i} = \vec{E}^a \times \vec{B}^a$. In this context, it is worth commenting on the two independent sources for 0^{++} states — the condensate, $\text{Tr}(FF)$, and the energy density, $T_{00} = \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a)$. Naively one might conclude the second one coming from the conserved energy momentum tensor should be dropped, for the same reason we dropped the operator for momentum conservation, T_{0i} . However in fact for QCD because of the conformal anomaly for the trace of the energy momentum tensor it is easy to show that the decoupling argument fails. A bag-like model circumvents the decoupling because the bag itself implies a scale breaking “vacuum” (empty bag) thus introducing an extra four-vector, the bag velocity u_μ . Our AdS black hole background, which is a key ingredient in our approach, also breaks conformal invariance. Consequently all agree that there is an extra low mass 0^{++} state in addition to the one which in our case is degenerate with the tensor 2^{++} . The lattice data clearly favors this low mass 0^{++} state, in agreement with our AdS spectrum. Finally, one rather low mass state, the 2^{-+} , is missing in the AdS spectrum. This state is clearly present in the lattice spectrum and is identified in the bag model.

At $d = 6$, we have two states identified in the $C = -1$ symmetric trace, $\text{Tr}(F_{\mu\nu}\{F_{\rho\sigma}, F_{\lambda\eta}\})$: the 1^{+-} state for operator $d^{abc}\vec{B}_a(\vec{E}_b \cdot \vec{E}_c)$ and the 1^{--} state for the operator $d^{abc}\vec{E}_a(\vec{E}_b \cdot \vec{E}_c)$. These states are clearly related to the field content of a IIA supersymmetric multiplet. The higher spin representation are not present at strong coupling. Moreover, we have no states corresponding to the antisymmetric trace operators, $\text{Tr}(F[F, F])$. It appears that in the limit where we restrict to supergravity modes and ignore the massive stringy states, we will be able to obtain only the glueball state with the symmetric d^{abc} coupling between the group indices, and not the state with the antisymmetric f^{abc} coupling. If we consider the chiral primaries of the 4-d Yang-Mills theory (which was the theory on the boundary before we took the τ direction to be compact), then we find that these had the form $\text{Sym Tr}(X^i X^j X^k)$ - i.e, we have a symmetric trace over the fields. Other operators that couple to the supergravity fields will be supersymmetry descendents of these chiral primaries, but the symmetry in the trace would be maintained¹. This fact may be related to the observation of Tseytlin [16] that generalizing the Born-Infeld action to a non-abelian case gives rise to symmetric trace operators in the field theory.

One legitimate point of view is simply to suppose that all missing states must by definition be stringy effects that will be restored in weak coupling. However we prefer to look on this as a possible clue on constructing a better initial geometry for the supergravity/QCD duality proposals.

¹We thank W. Taylor for a discussion on this point.

5.3 Strong coupling Expansion for Pomeron Intercept

We shall end this discussion with a comment on the slope of the leading glueball trajectory as way to estimate the crossover value for the bare coupling, where continuum physics might begin to hold. The Pomeron is the leading Regge trajectory passing through the lightest glueball state with $J^{PC} = 2^{++}$. In a linear approximation, it can be parameterized by

$$\alpha_P(t) = 2 + \alpha'_P(t - m_T^2), \quad (53)$$

where we can use the strong coupling estimate for the lightest tensor mass²,

$$m_T \simeq [9.86 + 0(\frac{1}{g^2 N})] \beta^{-1}. \quad (54)$$

Moreover if we make the standard assumption that the closed string tension is twice that between two static quark sources [19], we also have a strong coupling expression for the Pomeron slope,

$$\alpha'_P \simeq [\frac{27}{32\pi g^2 N} + 0(\frac{1}{g^4 N^2})] \beta^2. \quad (55)$$

Putting these together, we obtain a strong coupling expansion for the Pomeron intercept,

$$\alpha_P(0) \simeq 2 - 0.66 (\frac{4\pi}{g^2 N}) + 0(\frac{1}{g^4 N^2}). \quad (56)$$

Turning this argument around, we can estimate a crossover value between the strong and weak coupling regimes by fixing $\alpha_P(0) \simeq 1.2$ at its phenomenological value [17]. In fact this yields for QCD_4 at $N = 3$ a reasonable value for $\alpha_{strong} = g^2/4\pi = 0.176$ for the crossover. Much more experience with this new approach to strong coupling must be gained before such numerology can be taken seriously. However, similar crude argument have proven to be a useful guide in the crossover regime of lattice QCD. One might even follow the general strategy used in the lattice cut-off formulations. Postpone the difficult question of analytically solving the QCD string to find the true UV fixed point. Instead work at a fixed but physically reasonable cut-off scale (or bare coupling) to calculate the spectrum. If one is near enough to the fixed point, mass ratios should be reliable. After all, the real benefit of a weak/strong duality is to use each method in the domain where it provides the natural language. On the other hand, clearly from a fundamental point of view, finding analytical tools to understand the renormalized trajectory and prove asymptotic scaling within the context of the gauge invariant QCD string would also be a major achievement — an achievement that presumably would include a proof of confinement itself.

²We have adopted the normalization in the AdS -black hole metric to simplify the coefficients, e.g., for AdS^7 , $\bar{g}_{\tau\tau} = r^2 - r^{-4}$. This corresponds to fixing the “thermal-radius” $R_1 = 1/3$ so that $\beta = 2\pi R_1 = 2\pi/3$.

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A Wave Equations

In this appendix we outline the derivation of the wave equations that were used to find the energy levels in the supergravity theory. First we take the case of QCD_3 , for which we have the metric,

$$ds^2 = (r^2 - \frac{1}{r^2})d\tau^2 + r^2 \sum_{i=1,2,3} dx_i^2 + (r^2 - \frac{1}{r^2})^{-1}dr^2 + d\Omega_5^2, \quad (\text{A.1})$$

where x_3 is the Euclidean time direction.

The simplest equation is the scalar wave equation for the dilaton and the axion. At the linear perturbation level, both satisfy

$$\phi_{,\mu}{}^{;\mu} = \frac{1}{\sqrt{-g}} [\phi_{,\mu} g^{\mu\nu} \sqrt{-g}]_{,\nu} = 0. \quad (\text{A.2})$$

We introduce a plane wave ansatz,

$$\phi = T_3(r) e^{ik_3 x_3}, \quad (\text{A.3})$$

with zero momentum and mass, $m = ik_3$ providing the equation (8) for T_3 in the text.

Fluctuations in the volume of the sphere S^5 , which is a fiber at each point of space time, provides another scalar mode. This scalar has an AdS mass squared equal to 32. The field equation is

$$\frac{1}{\sqrt{-g}} [\phi_{,\mu} g^{\mu\nu} \sqrt{-g}]_{,\nu} - 32 \phi = 0, \quad (\text{A.4})$$

$\phi = L_3(r) e^{ik_3 x_3}$, which reduces to the equation (8) for L_3 in the text.

Next let us address the case of the two-form fields, $B_{\mu\nu}$ and $C_{\mu\nu}$, which at the linear level are conveniently combined into a single complex field, $\tilde{B}_{\mu\nu} = B_{\mu\nu} + iC_{\mu\nu}$. It was shown in [27] that this field satisfies the equation,

$$(\text{Max} - k(k+4))\tilde{B}_{\mu\nu} + 2i\epsilon_{\mu\nu}{}^{\rho\lambda\sigma} \partial_\rho \tilde{B}_{\lambda\sigma} = 0, \quad (\text{A.5})$$

for $k = 0, 1, \dots$ harmonics on S^5 . The Maxwell operator is defined by

$$\text{Max} \tilde{B}_{\mu\nu} = H_{\mu\nu\lambda}{}^{;\lambda}. \quad (\text{A.6})$$

in terms of the field strength,

$$H_{\mu\nu\lambda} = \partial_\mu \tilde{B}_{\nu\lambda} + \partial_\nu \tilde{B}_{\lambda\mu} + \partial_\lambda \tilde{B}_{\mu\nu}. \quad (\text{A.7})$$

Since the second order differential operator factorizes into two first order operators, solutions fall into two classes,

$$(2kI + i * D)\tilde{B}_{\mu\nu}^{(1)} = 0, \quad (2(k+4)I - i * D)\tilde{B}_{\mu\nu}^{(2)} = 0, \quad (\text{A.8})$$

where $(*DB)_{\mu\nu} = \epsilon_{\mu\nu}^{\rho\lambda\sigma} \partial_\rho \tilde{B}_{\lambda\sigma}$ and I is the identity matrix.

It is convenient to iterate these first order equations to get the second order equations,

$$(\text{Max} - k^2)\tilde{B}_{\mu\nu}^{(1)} = 0, \quad (\text{Max} - (k+4)^2)\tilde{B}_{\mu\nu}^{(2)} = 0. \quad (\text{A.9})$$

We are interested in fields with no dependence on the coordinates of the sphere, so we can take $k = 0$. It can be shown that the first class of solutions, $\tilde{B}_{\mu\nu}^{(1)}$, are pure gauge, so we are only interested in the second class, which has an effective mass squared of 16 for the field $\tilde{B}_{\mu\nu}^{(2)}$.

As explained in the text, one must of course check that solutions to the second order equation for $\tilde{B}_{\mu\nu}^{(2)}$, actually are valid solution to the original wave equation. This reduce the number of independent tensor fields, $\tilde{B}_{\mu\nu}$, from 6 to 3. For example with the ansatz,

$$\tilde{B}_{12} = N_3(r)r^2 e^{ik_3 x_3} \quad (\text{A.10})$$

the first order equations will determine $\tilde{B}_{\tau 3}$ and $\tilde{B}_{r\tau}$ once we have a solution of the second order equation for \tilde{B}_{12} . This does not place any constraints on the solution for \tilde{B}_{12} itself. We have defined \tilde{B}_{12} in terms of the normalized coefficient $N_3(r)$ to obtain a hermitian field equation for N_3 similar to our earlier scalar mode T_3 . In a similar manner we can find the equation for $\tilde{B}_{1\tau}$. We can solve the second order equation for this fluctuation without constraint, and then the requirement arising from the associated first order equations determines corresponding values of \tilde{B}_{23} and \tilde{B}_{2r} . After adopting the ansatz indicated in the main text, we again obtain a hermitian equation for $M_3(r)$.

The graviton perturbations arise from the Einstein action with a cosmological constant, expanded around the given background. We write $G_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. the equation for the perturbation $h_{\mu\nu}$ is

$$-\frac{1}{2}h_{\mu\nu;\lambda}{}^{;\lambda} - \frac{1}{2}h_{\lambda;\mu\nu}^\lambda + \frac{1}{2}h_{\mu\lambda;\nu}{}^{;\lambda} + \frac{1}{2}h_{\nu\lambda;\mu}{}^{;\lambda} + 4h_{\mu\nu} = 0 \quad (\text{A.11})$$

Near the boundary at $r = \infty$, we can choose a gauge to make the perturbations transverse to r, x_3 and traceless. It turns out that we can maintain this condition for all r for perturbations of the form h_{12} , and for perturbations of the form $h_{1\tau}$. In these cases the above equation for $h_{\mu\nu}$ gives immediately the wave equations to be solved. But keeping in mind the decomposition in spin eigenstates in the $x_1 - x_2$ plane, we also find that we have to

consider a spin-0 perturbation which at infinity has the form $h_{\tau\tau}$, with $h_{11} = h_{22} = -\frac{1}{2}h_{\tau\tau}$. In this case we can choose a gauge to make $h_{3\mu} = 0$, but for finite r we will find in this gauge that $h_{rr} \neq 0$ and also that the part transverse to r, x_3 is not traceless. Thus we have to keep $h_{\tau\tau}, h_{11} = h_{22}, h_{rr}$ as independent coupled functions in the analysis. It was shown in [10] how these equations can be reduced to one effective equation which can then be solved in the same way as the equations for the other fields. A nicer choice of gauge was used in [12] which led to an equivalent but simpler equation. We will use the latter source for the equations, especially since the results there include all dimensions, and so can be used for the case of QCD_4 as well.

Now we turn to the case of $\mathbf{AdS}^7 \times \mathbf{S}^4$, which is very similar. The metric is

$$ds^2 = (r^2 - \frac{1}{r^4})d\tau^2 + r^2 \sum_{i=1,2,3,4,11} dx_i^2 + (r^2 - \frac{1}{r^4})^{-1}dr^2 + \frac{1}{4}d\Omega_4^2, \quad (\text{A.12})$$

where we have x_4 as the time direction and we note that the radius of the S^4 is half the curvature radius of the AdS space: this will affect the masses arising from the deformations of the sphere.

There is a three-form field $A_{\mu\nu\lambda}$ which behaves in a manner similar to the two-form field $\tilde{B}_{\mu\nu}$ discussed above. Its field equation can again be factorized into two first order equations, which we iterated to second order equations. One factor at $k = 0$ corresponds to pure gauge, while the other at $k = 0$ has the value $m_{AdS}^2 = 4(k+3)^2 = 36$. For the scalar mode due to fluctuations of the volume of the S^4 , we get a $m_{AdS}^2 = 4 \times 18 = 72$. For these considerations we arrive at the wave equations (22) given in the text.

Finally for comparison with P. van Nieuwenhuizen Ref [28], we note that they have scaled the radius of the S^4 to unity, instead of $\frac{1}{2}$. Our choice was made to keep the radius of the AdS^7 equal to unity, to make the comparison between AdS^5 and AdS^7 more natural. This change in metric scales the squared masses by a factor of 4 relative to Ref. [28].

B WKB and Variational Estimates

First we change variable from r to $x \equiv r^2$ and express all twelve equations in the standard Sturm-Liouville form,

$$\left\{ -\frac{d}{dx}\tau(x)\frac{d}{dx} + w(x) \right\} \phi_n(x) = m_n^2 \sigma(x)\phi_n(x),$$

where $\tau(x)$, $w(x)$, and $\sigma(x)$ are generalized “tension”, “external force”, and “mass-density” respectively. In our case, we have for QCD_4 : $\tau_4(x) = (x^4 - x)$ and $\sigma_4(x) = \frac{x}{4}$; for QCD_3 : $\tau_3(x) = (x^3 - x)$ and $\sigma_3(x) = \frac{1}{4}$. Force densities $w(x)$ for all 12 cases are listed in Table 5. We shall use this as our starting point for carrying out variational and WKB analyses.

B.1 Variational Estimates for m_0^2 :

Solving for eigenstates, $\{\phi_n(x)\}$ and their corresponding eigenvalues, $\{m_n^2\}$, is equivalent to finding stationery points of the following functional,

$$\Gamma[\phi] \equiv \frac{\int_1^\infty dx [\tau(x)\phi'(x)^2 + w(x)\phi(x)^2]}{\int_1^\infty dx \sigma(x)\phi(x)^2}, \quad (\text{B.1})$$

with $m_n^2 = \Gamma[\phi_n]$. In particular, the square of the mass for the lowest state, $m_0^2 = \Gamma[\phi_0]$, is the absolute minimum of $\Gamma[\phi]$.

To be properly defined as a Sturm-Liouville problem, it is necessary to impose boundary conditions: $\tau(x)\phi(x)\phi'(x) \rightarrow 0$, for $x \rightarrow 1$ and for $x \rightarrow \infty$, in accord with the boundary conditions stated earlier for our numerical solutions. Explicit limiting behaviors for all 12 cases are listed in Table 5.

Given a trial wave function, $\phi(x)$, Eq. B.1 provides a variational upper bound for m_0^2 . As we have shown in Ref. [10], accurate variational estimates for ground-state masses can be obtained with minimum efforts. Here, we shall only attempt to obtain simple estimates by choosing trial functions so that integrals in the equation above can be evaluated analytically.

The simplest possible trial wave function for each case can be chosen as a product of $\tau(x)$ and x^{-1} , as indicated in Table 5. Indeed, our variational approach has served us well by providing a useful consistency check for our numerical efforts along the way. These are also summarized in Table 5.

B.2 WKB

As explained in Ref.[10], we begin a WKB analysis by first bringing our differential equations from the Sturm-Liouville form into a radial-Schroedinger form ,

$$\left(-\frac{d^2}{dx^2} + V(x; m^2)\right)\psi(x) = E\psi(x), \quad (\text{B.2})$$

Equation	$w(x)$	$x \rightarrow \infty$	$x \rightarrow 1$	Trial Function	Bound for m_0^2
T_4	0	x^{-3}	1	x^{-3}	24.0
V_4	$\frac{9}{4}x^{-1}(x^3 - 1)^{-1}$	x^{-3}	$(x - 1)^{1/2}$	$(x^3 - 1)^{1/2}x^{-9/2}$	37.3
S_4	$-108x^2(5x^3 - 2)^{-2}$	x^{-3}	1	x^{-3}	8.16
N_4	$\frac{9}{4}(3x^3 - 1)x^{-1}$	$x^{-9/2}$	1	$x^{-9/2}$	56.0
M_4	$\frac{9}{4}x^2(3x^3 - 2)(x^3 - 1)^{-1}$	$x^{-9/2}$	$(x - 1)^{1/2}$	$(x^3 - 1)^{1/2}x^{-6}$	93.3
L_4	$18x^2$	x^{-6}	1	x^{-6}	120
T_3	0	x^{-2}	1	x^{-2}	12.0
V_3	$x^{-1}(x^2 - 1)^{-1}$	x^{-2}	$(x - 1)^{1/2}$	$(x^2 - 1)^{1/2}x^{-3}$	20.0
S_3	$-16x(3x^2 - 1)^{-2}$	x^{-2}	1	x^{-2}	5.55
N_3	$(3x^2 - 1)x^{-1}$	x^{-2}	1	x^{-2}	27.0
M_3	$x(3x^2 - 2)(x^2 - 1)^{-1}$	x^{-3}	$(x - 1)^{1/2}$	$(x^2 - 1)^{1/2}x^{-4}$	46.7
L_3	$8x$	x^{-4}	1	x^{-4}	56.0

Table 5: Variational Estimates for QCD_4 and QCD_3

by scaling $\psi(x) \equiv \sqrt{\tau(x)}\phi(x)$. Eigenvalues m_n^2 are found by solving for zero-energy bound states from below, $E \rightarrow 0^-$. The potential $V(x; m^2)$ is given by

$$V(x; m^2) = \frac{-m^2\sigma(x) + w(x)}{\tau(x)} + \frac{1}{2}\left(\frac{\tau}{\tau}\right)' - \frac{1}{4}\left(\frac{\tau'}{\tau}\right)^2. \quad (\text{B.3})$$

To obtain the desired WKB estimate, we simply need to evaluate in the large m^2 limit the following integral: $(n + 1/2)\pi = \int_1^\infty dx \sqrt{-\tilde{V}(x; m^2)}$, i.e., we seek a WKB condition in the form, $(n + \frac{1}{2})\pi = s_0 m + s_1 + 0(\frac{1}{m})$, where coefficients s_0 and s_1 can be explicitly evaluated. As explained in Ref. [10], the “effective potential”, $\tilde{V}(x, m^2)$, is $\tilde{V}(x; m^2) = V(x; m^2) + 1/(4(x - 1)^2)$.

To isolate the m^2 -dependence, let us write $\tilde{V}(x, m^2) \equiv m^2 V_0(x) + \tilde{V}_1(x)$, where $V_0(x) = -\sigma(x)/\tau(x)$, i.e., $V_0^{(4)}(x) = -1/(4(x^3 - 1))$, and $V_0^{(3)}(x) = -1/(4x(x^2 - 1))$. For instance, one immediately finds that $s_0 = \int_1^\infty dx \sqrt{-V_0(x)}$. For QCD_4 and QCD_3 , they are $s_0^{(4)} = \frac{\sqrt{\pi}}{6}\Gamma(\frac{1}{6})/\Gamma(\frac{2}{3})$, and $s_0^{(3)} = \frac{\sqrt{\pi}}{4}\Gamma(\frac{1}{4})/\Gamma(\frac{3}{4})$. The remaining piece, $\tilde{V}_1(x)$, is listed in Table 6.

To find coefficient s_1 , we need to know the behavior of the ratio of $V_0(x)$ to $\tilde{V}_1(x)$ near $x \rightarrow 1$ and $x \rightarrow \infty$. The dominant behavior of $\tilde{V}_1(x)$, in these limits, can be characterized by two indices,

$$\tilde{V}_1(x) \sim \frac{\delta_l^2}{4(x - 1)^2} + 0\left(\frac{1}{(x - 1)}\right), \quad \text{for } x \rightarrow 1, \quad (\text{B.4})$$

$$\tilde{V}_1(x) \sim \frac{\delta_r^2}{4x^2} + 0\left(\frac{1}{x^3}\right), \quad \text{for } x \rightarrow \infty. \quad (\text{B.5})$$

Equation #	Effective Potential $\tilde{V}_1(x)$	δ_l	δ_r	δ	γ
T_4	$\frac{1}{4(x-1)^2} + \frac{2}{x^2} - \frac{9}{4x^2(x^3-1)^2}$	0	3	4	3.31
V_4	$\frac{1}{4(x-1)^2} + \frac{2}{x^2}$	1	3	5	4.90
S_4	$\frac{1}{4(x-1)^2} + \frac{2}{x^2} - \frac{9}{4x^2(x^3-1)^2} - \frac{108x}{(x^3-1)(5x^3-2)^2}$	0	3	4	2.20
N_4	$\frac{1}{4(x-1)^2} + \frac{2}{x^2} - \frac{9}{4x^2(x^3-1)^2} + \frac{9(3x^3-1)}{4x^2(x^3-1)}$	0	6	7	8.50
M_4	$\frac{1}{4(x-1)^2} + \frac{2}{x^2} + \frac{9(3x^3+1)}{4x^2(x^3-1)}$	1	6	8	12.5
L_4	$\frac{1}{4(x-1)^2} + \frac{2}{x^2} - \frac{9}{4x^2(x^3-1)^2} + \frac{18x}{x^3-1}$	0	9	10	17.5
T_3	$\frac{1}{4(x-1)^2} + \frac{3}{4x^2} - \frac{1}{x^2(x^2-1)^2}$	0	2	3	2.01
V_3	$\frac{1}{4(x-1)^2} + \frac{3}{4x^2}$	1	2	4	3.28
S_3	$\frac{1}{4(x-1)^2} + \frac{3}{4x^2} - \frac{1}{x^2(x^2-1)^2} - \frac{16}{(x^2-1)(3x^2-1)^2}$	0	2	3	-3.70
N_3	$\frac{1}{4(x-1)^2} + \frac{3}{4x^2} - \frac{1}{x^2(x^2-1)^2} + \frac{3x^2-1}{x^2(x^2-1)}$	0	4	5	4.71
M_3	$\frac{1}{4(x-1)^2} + \frac{3}{4x^2} + \frac{3x+1}{x^2(x+1)^2(x-1)}$	1	4	6	7.66
L_3	$\frac{1}{4(x-1)^2} + \frac{3}{4x^2} - \frac{1}{x^2(x^2-1)^2} + \frac{8}{x^2-1}$	0	6	7	9.50

Table 6: WKB indices for QCD_4 and QCD_3

One finds that $s_1 = -(\frac{\pi}{2})(\delta_l + \delta_r)$, and $m^2 = (\frac{\pi}{s_0})^2\{n^2 + \delta n + \gamma\} + 0(n^{-1})$ where

$$\delta = 1 + \delta_l + \delta_r, \quad (\text{B.6})$$

as exhibited in Table 6. The values for γ are found by a fit to the numerical values of m_n^2 .

References

- [1] J. Maldacena, “The Large N limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. 2:231, 1998, hep-th/9711200.
- [2] E. Witten, “Anti-de Sitter space and holography”, Adv. Theor. Math. Phys. 2: 253, 1998, hep-th/9805028.
- [3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “Gauge theory correlators from non-critical string theory”, Phys. Lett. **B428** (1998) 105, hep-th/9802109.
- [4] E. Witten, “Anti-de Sitter space, Thermal Phase Transition, and Confinement in Gauge Theories”, Adv. Theor. Math. Phys. 2: 505, 1998, hep-th/9803131.
- [5] C. J. Morningstar and M. Peardon, “The glueball spectrum from an anisotropic lattice study”, Phys.Rev. D60 (1999) 034509.
- [6] C. Csáki, H. Ooguri, Y. Oz and J. Terning, “Glueball Mass Spectrum From Supergravity”, hep-th/9806021.
- [7] R. De Mello Koch, A. Jevicki, M. Mihailescu and J. Nunes, “Evaluation of glueball masses from supergravity,” hep-th/9806125.
- [8] D. Gross and H. Ooguri, Phys. Rev. **D58** (1988) 106002, hep-th/9805129, M. Zyskin, Phys. Lett. **B439** (1998) 373, hep-th/9806128, H. Ooguri, H. Robins and J. Tannenhauser, Phys. Lett. **B437** (1998) 77, hep-th/9806171, J.G. Russo, hep-th/9808117, A. Hashimoto and Y. Oz, hep-th/9809106, C. Csáki, H. Ooguri, Y. Oz and J. Terning, hep-th/9810186, S.S. Gubser, hep-th/9810225, P. Kraus, F. Larsen and S.P. Trivedi, hep-th/9811120, R.G. Cai and K.S. Soh, hep-th/9812121, J.G. Russo and K. Sfetsos, hep-th/9901056, C. Csáki, J. Russo, K. Sfetsos and J. Terning, hep-th/9902067.
- [9] A. Hashimoto and Y. Oz, hep-th/9809106 v3.
- [10] R.C. Brower, S. Mathur and C-I Tan, hep-th/9908196, Nucl. Phys., in press.
- [11] R.C. Brower, S. Mathur and C-I Tan, hep-lat/9911030, Nucl. Phys. Proc. Suppl., in press; Proc. for “1999 Int. Symposium on Multiparticle Production”, World Scientific, in press.
- [12] N. R. Constable and R. C. Myers, hep-th/9908175.
- [13] J. Minahan, “Glueball mass spectra and other issues for supergravity duals of QCD models,” hep-th/9811156.
- [14] O. DeWolfe, D.Z. Freedman, S.S. Gubser and A. Karch, “Modeling the fifth dimension with scalars and gravity”, hep-th/9909134.

- [15] A. Harte and O. Philipsen, “The spectrum of the three-dimensional adjoint Higgs model and hot SU(2) gauge theory”, hep-lat/9908041.
- [16] A.A. Tseytlin “On non-abelian generalization of Born-Infeld action in string theory” Nucl.Phys. B501 (1997) 41-52.
- [17] C-I Tan, “Diffractive Production at Collider Energies and Factorization”, Phys. Report, **315** (1999) 175, hep-ph/9706276, hep-ph/9810237.
- [18] A. Capella, U. Sukhatme, C-I Tan, and J.T.V. Tran, Phys. Report, **236** (1994) 225.
- [19] A. Brandhuber, N. Itzhaki, J. Sonnenschein, and S. Yankielowicz, hep-th/98032263 v3.
- [20] S. Chandrasekhar, ‘The Mathematical theory of Black Holes’ (1983) *Oxford University Press*.
- [21] T. Regge and J. A. Wheeler, Phys. Rev. **108** (1957) 1063.
- [22] H. Fritzsch and P. Minkowski, Nuovo Cim., 30A (1975) 393.
- [23] J.D. Bjorken, 1979 SLAC Summer Institute, SLAC-PUB-2372 (1979).
- [24] R.L. Jaffe, K. Johnson, Z. Ryzak, “Qualitative Features of the Glueball Spectrum”, Ann. Phys. 168 (1986) 344.
- [25] For a recent attempt, see: A.B. Kaidalov and Yu.A. Simonov. hep-ph/9911291.
- [26] J. Kuti, Nucl. Phys. Proc. Suppl. 73 (1999) 590-595 and private communication (1999).
- [27] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, Phys. Rev. **D32** (1985) 389.
- [28] P. van Nieuwenhuizen, Class. and Quant. Grav. **2** (1985) 1.